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Wiener, hyper-Wiener, detour and hyper-detour indices of bridge and chain graphs

Toufik Mansour · Matthias Schork

Abstract Given a collection of connected graphs one may build bridge and chain graphs out of them. In this paper it is shown how the Wiener, hyper-Wiener, detour and hyper-detour indices for bridge and chain graphs are determined from the respective indices of the individual graphs. The results obtained are illustrated by some examples.

Keywords Wiener index · Hyper-Wiener index · Detour index · Hyper-detour index · Bridge graph · Chain graph

1 Introduction

Like in many other branches of mathematics one tries to find in graph theory certain invariants of graphs which only depend on the graph G itself (or—in other cases—in addition on an embedding into the plane or some other manifold), see, e.g., [6] and the references given therein. One of the simplest such invariants is the *adjacency matrix* $A(G)$ with columns and rows indexed by the vertices of the graph, such that the uv-entry of the matrix is equal to the number of arcs between the vertices u and v. The spectrum of the adjacency matrix is called the *spectrum of the graph* and the dependance of the spectrum from the graph has been considered in great detail, see, e.g., [4] and the references given therein. If $\sigma(G) \equiv \sigma(A(G)) = \{\lambda_1(G), \ldots, \lambda_n(G)\}$ is the spectrum of G then any function of the eigenvalues may be used as graph invariant.

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Of course, it is possible to define many other graph invariants which are not based on the spectrum of the adjacency matrix but directly on the vertices and edges of the graph and the distances between them. For example, in acyclic graphs the *Wiener index* is defined as $W(G) = \sum_{e \in E(G)} N_{i,e} N_{j,e}$, where $E(G)$ denotes the set of edges of G and $N_{k,e}$ denotes the number of vertices lying on the side of the edge e having the endpoint k [5,8,26]. Arguably, the Wiener index is the most famous such index but many other related indices have been considered, see, e.g., [5]. The motivation for the study of these objects is twofold: on the one hand one has the purely mathematical desire to understand the intrinsic properties of graphs and the connections between them better while on the other hand these indices are applied in theoretical chemistry as so called topological indices, capturing some of the properties of a molecule in a single number. More precisely, a single number, representing a chemical structure in graph-theoretical terms via the molecular graph, is called a *topological descriptor* and if it in addition correlates with a molecular property it is called *topological index*. By now there do exist a lot of different types of such indices which capture different aspects of the molecular graphs associated to the molecules considered, see, e.g., [5]. As already mentioned above the most famous such index is arguably the Wiener index [8,26]. The Szeged index [7,13] is closely related to the Wiener index and is a vertex-multiplicative type index that takes into account how the vertices of a given molecular graph are distributed. The Padmakar-Ivan (PI) index [12,14] is an additive index that takes into account the distribution of edges and, therefore, complements the Szeged index in a certain sense. For bridge and chain graphs (to be defined more precisely later) the PI index was determined in [19] and for bridge graphs the Szeged index (and the vertex PI index) was considered in [20]. Quite recently, the Wiener index was considered in [2] for a class of graphs containing as special cases the bridge and chain graphs. In the present paper we will derive the Wiener, hyper-Wiener, detour and hyper-detour indices of bridge and chain graphs in an explicit fashion. To make this more concrete we now define these indices.

In an undirected connected graph G with vertices set $V(G)$ and edges set $E(G)$, a given pair of vertices (a, b), $a, b \in V(G)$, is joined by a path $p(a, b)$, that is, a continuous sequence of edges, with the property that all distinct and any two subsequent edges are adjacent, see [10,23]. The length of the path $p(a, b)$ is equal to the length of edges in the path between vertices a and b. The shortest path joining vertices a and b is called *geodesic* and its length is the topological distance, D_{ab}. The longest path is the *elongation* and its length is equal to the detour distance, Δ_{ab}. The square arrays which collect the lengths of the two path types are called the *distance matrix* (see [3,10,23]), denoted as **D**, and the *detour matrix* (see [1,3,11,24]), denoted as **Δ**, respectively:

$$\mathbf{D}_{ab} = \begin{cases} D_{ab}, & a \neq b \\ 0, & a = b, \end{cases} \quad \mathbf{\Delta}_{ab} = \begin{cases} \Delta_{ab}, & a \neq b \\ 0, & a = b. \end{cases} \tag{1.1}$$

In the following we will adopt this convention and do not indicate the dependency on the graph explicitly (i.e., we will write $\mathbf{\Delta}_{ab}$ instead of $\mathbf{\Delta}_{ab}(G)$).

Several graph descriptors (topological indices) can be calculated as half-sum of entries in the above matrices: The half-sum of all the entries of the matrices **D** and **Δ**

are called the *Wiener index* (see [26]) and the *Detour index* (see [1,11,17,18,22,24,25])
of the graph G and they are denoted by $W = W(G)$ and $\omega = \omega(G)$, that is,

$$W = \frac{1}{2} \sum_{a,b} \mathbf{D}_{ab}, \quad \omega = \frac{1}{2} \sum_{a,b} \mathbf{\Delta}_{ab}, \tag{1.2}$$

respectively. The hyper-Wiener index (see [9,16,21]) and the hyper-detour index (see
[15,17]) are given by

$$WW = \frac{1}{2} \sum_{a,b} \binom{\mathbf{D}_{ab} + 1}{2}, \quad \omega\omega = \frac{1}{2} \sum_{a,b} \binom{\mathbf{\Delta}_{ab} + 1}{2}. \tag{1.3}$$

In the present paper we will consider bridge and chain graphs and determine the
Wiener, hyper-Wiener, detour, and hyper-detour indices for them. The important spe-
cial case where the bridge and chain graphs are built from several copies of the same
graph is singled out and the asymptotics of the above indices for large graphs are given
in this case. Furthermore, some particular examples are treated explicitly.

The paper is organized as follows. In Sect. 2 bridge and chain graphs are defined
and some important quantities are introduced which will be used repeatedly in the sec-
tions following. In Sect. 3 the Wiener and detour indices of bridge graphs are treated,
whereas the hyper-Wiener and hyper-detour indices of bridge graphs are treated in
Sect. 4. Similarly, in Sect. 5 the Wiener and detour indices of chain graphs are treated,
whereas the hyper-Wiener and hyper-deotur indices of chain graphs are treated in
Sect. 6. Finally, in Sect. 7 some conclusions are presented.

2 Preliminaries: bridge and chain graphs

In this section we introduce bridge and chain graphs and establish some notations and
abbreviations which will be used in the rest of the article. We also collect here some
simple observations which will be used repeatedly in later sections.

2.1 Bridge graphs

Let $\{G_i\}_{i=1}^d$ be a set of finite pairwise disjoint graphs with $v_i \in V(G_i)$. The *bridge
graph*

$$B(G_1, G_2, \ldots, G_d) \equiv B(G_1, G_2, \ldots, G_d; v_1, v_2, \ldots, v_d)$$

of $\{G_i\}_{i=1}^d$ with respect to the vertices $\{v_i\}_{i=1}^d$ is the graph obtained from the graphs
$G_1, \ldots, G_d$ by connecting the vertices v_i and v_{i+1} by an edge for all $i = 1, 2, \ldots,$
$d - 1$, see Fig. 1.

Fig. 1 The bridge graph
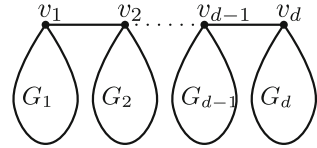
In the following we will define several quantities which will be used repeatedly later on. As a first quantity we introduce

$$P_b(G) := \sum_{a \in V(G)} \mathbf{D}_{ab}, \quad Q_b(G) := \sum_{a \in V(G)} \boldsymbol{\Delta}_{ab}, \tag{2.1}$$

which is, roughly speaking, half the contribution of the vertex b to the Wiener index (resp. detour index) of G. More precisely, using (1.2), we can write

$$W(G) = \frac{1}{2} \sum_{b \in V(G)} P_b(G), \quad \omega(G) = \frac{1}{2} \sum_{b \in V(G)} Q_b(G).$$

Similarly, we introduce the analogous quantities

$$\widetilde{P}_b(G) := \sum_{a \in V(G)} \binom{\mathbf{D}_{ab} + 1}{2}, \quad \widetilde{Q}_b(G) := \sum_{a \in V(G)} \binom{\boldsymbol{\Delta}_{ab} + 1}{2} \tag{2.2}$$

for the hyper-Wiener and hyper-detour index. Recalling (1.3), we can write

$$WW(G) = \frac{1}{2} \sum_{b \in V(G)} \widetilde{P}_b(G), \quad \omega\omega(G) = \frac{1}{2} \sum_{b \in V(G)} \widetilde{Q}_b(G).$$

Since we will have to consider later the shortest and longest paths between two vertices in a bridge graph we collect some obvious facts here for easy reference.

Fact 2.1 *Let $B(G_1, \ldots, G_d; v_1, \ldots, v_d)$ be the bridge graph of $\{G_i\}_{i=1}^d$ and let two vertices $a \in V(G_i)$ and $b \in V(G_j)$ with $i < j$ be given. Then the following holds true.*

(1) *Each shortest path $p(a, b)$ in G between a and b can be decomposed into three shortest paths: the shortest path $p(a, v_i)$ in the graph G_i, the path $v_i v_{i+1} \ldots v_j$, and the shortest path $p(v_j, b)$ in the graph G_j. Thus, we can write*

$$\mathbf{D}_{ab} = \mathbf{D}_{av_i} + (j - i) + \mathbf{D}_{bv_j}.$$

(2) *Each longest path $p(a, b)$ in G between a and b can be decomposed into the following paths: a longest path $p(a, v_i)$ in the graph G_i, the path $v_r v_{r+1}$ (of length 1) and the longest path in the graph G_{r+1} which starts and ends at v_{r+1} for all $r = i, \ldots, j-1$, and the longest path $p(v_j, b)$ in the graph G_j. Denoting*

Fig. 2 The chain graph
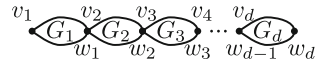
the length of the longest path in the graph G_s which starts and ends at v_s by c_{v_s}, we can thus write

$$\mathbf{\Delta}_{ab} = \mathbf{\Delta}_{av_i} + \sum_{s=i+1}^{j-1} (1 + c_{v_s}) + 1 + \mathbf{\Delta}_{bv_j}.$$

2.2 Chain graphs

Let $\{G_i\}_{i=1}^{d}$ be a set of finite pairwise disjoint graphs with $v_i, w_i \in V(G_i)$, the *chain graph*

$$C(G_1, G_2, \ldots, G_d) \equiv C(G_1, G_2, \ldots, G_d; v_1, w_1, v_2, w_2, \ldots, v_d, w_d)$$

of $\{G_i\}_{i=1}^{d}$ with respect to the vertices $\{v_i, w_i\}_{i=1}^{d}$ is the graph obtained from the graphs $G_1, \ldots, G_d$ by identifying the vertex w_i and the vertex v_{i+1} for all $i = 1, 2, \ldots, d-1$, see Fig. 2.

Given a chain graph as in Fig. 2, we define in addition to the above quantities

$$L_P(G_i) := P_{v_i}(G_i) \sum_{j=1}^{i-1} (|V(G_j)| - 1), \quad R_P(G_i) := P_{v_{i+1}}(G_i) \sum_{j=i+1}^{d} (|V(G_j)| - 1)$$

$$(2.3)$$

where $L_P(G_i)$ is defined for $i = 2, \ldots, d$ and $R_P(G_i)$ for $i = 1, \ldots, d - 1$. Note that the letter L is chosen as a memnonic for "left" (and similarly for R). Similarly,

$$L_Q(G_i) := Q_{v_i}(G_i) \sum_{j=1}^{i-1} (|V(G_j)| - 1), \quad R_Q(G_i) := Q_{v_{i+1}}(G_i) \sum_{j=i+1}^{d} (|V(G_j)| - 1)$$

$$(2.4)$$

and $L_Q(G_i)$ is defined for $i = 2, \ldots, d$ and $R_Q(G_i)$ for $i = 1, \ldots, d - 1$. Roughly speaking, these quantities represent some kind of weighted contribution of one side of the considered vertex to the respective index. For example, in $L_P(G_i)$ the contribution $P_{v_i}(G_i)$ to the Wiener index is weighted with the sum of the vertices to the left of v_i, whereas in $R_Q(G_i)$ the contribution $Q_{v_{i+1}}(G_i)$ to the detour index is weighted with the sum of the vertices to the right of $v_{i+1} = w_i$. We also introduce the corresponding quantities for the hyper-Wiener and hyper-detour index,

$$\widetilde{L}_P(G_i) := \widetilde{P}_{v_i}(G_i) \sum_{j=1}^{i-1}(|V(G_j)| - 1), \quad \widetilde{R}_P(G_i) := \widetilde{P}_{v_{i+1}}(G_i) \sum_{j=i+1}^{d}(|V(G_j)| - 1),$$

$$\text{(2.5)}$$

as well as

$$\widetilde{L}_Q(G_i) := \widetilde{Q}_{v_i}(G_i) \sum_{j=1}^{i-1}(|V(G_j)| - 1), \quad \widetilde{R}_Q(G_i) := \widetilde{Q}_{v_{i+1}}(G_i) \sum_{j=i+1}^{d}(|V(G_j)| - 1).$$

$$\text{(2.6)}$$

In analogy to Fact 2.1 one has in the case of chain graphs the following fact.

Fact 2.2 *Let $C(G_1, \ldots, G_d; v_1, w_1, \ldots, v_d, w_d)$ be the chain graph of $\{G_i\}_{i=1}^{d}$ and let two vertices $a \in V(G_i)$ and $b \in V(G_j)$ with $i < j$ be given. Then the following holds true.*

(1) *Each shortest path $p(a, b)$ in G between a and b can be decomposed into three shortest paths: the shortest path $p(a, w_i)$ in the graph G_i, the shortest path from v_s to w_s in the graph G_s for $s = i + 1, \ldots, j - 1$, and the shortest path $p(v_j, b)$ in the graph G_j. Thus, we can write*

$$\mathbf{D}_{ab} = \mathbf{D}_{aw_i} + \sum_{j=i+1}^{j-1} \mathbf{D}_{v_s w_s} + \mathbf{D}_{bv_j}.$$

(2) *Each longest path $p(a, b)$ in G between a and b can be decomposed into the following paths: the longest path $p(a, w_i)$ in the graph G_i, the longest path from v_s to w_s in the graph G_s for $s = i + 1, \ldots, j - 1$, and the longest path $p(v_j, b)$ in the graph G_j. Thus, we can write*

$$\mathbf{\Delta}_{ab} = \mathbf{\Delta}_{aw_i} + \sum_{j=i+1}^{j-1} \mathbf{\Delta}_{v_s w_s} + \mathbf{\Delta}_{bv_j}.$$

3 Wiener and detour indices of the bridge graph

In this section we give a formula for the Wiener and detour indices of the bridge graph $B(G_1, G_2, \ldots, G_d)$ in terms of the graphs G_i.

3.1 Wiener index of the bridge graph

At first we consider the Wiener index for the bridge graph.

Theorem 3.1 *The Wiener index of the bridge graph $G = B(G_1, \ldots, G_d; v_1, \ldots, v_d)$ is given by*

$$W(G) = \sum_{i=1}^{d} W(G_i)$$

$$+ \sum_{1 \le i < j \le d} |V(G_i)||V(G_j)|(j-i) + \sum_{i=1}^{d} (|V(G)| - |V(G_i)|) P_{v_i}(G_i).$$

Proof From the definitions we have that

$$2W(G) = \sum_{a,b \in V(G)} \mathbf{D}_{ab}$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{a \in V(G_i)} \sum_{b \in V(G_j)} \mathbf{D}_{ab}$$

$$= \sum_{i=1}^{d} \sum_{a,b \in V(G_i)} \mathbf{D}_{ab} + \sum_{i=1}^{d} \sum_{j=1, j \ne i}^{d} \sum_{a \in V(G_i)} \sum_{b \in V(G_j)} \mathbf{D}_{ab}$$

$$= 2 \sum_{i=1}^{d} W(G_i) + 2 \sum_{1 \le i < j \le d} \sum_{a \in V(G_i)} \sum_{b \in V(G_j)} \mathbf{D}_{ab}.$$

Recalling Fact 2.1 about the decomposition of the shortest path in a bridge graph, we can thus write

$$2W(G) = 2 \sum_{i=1}^{d} W(G_i) + 2 \sum_{1 \le i < j \le d} \sum_{a \in V(G_i)} \sum_{b \in V(G_j)} \{\mathbf{D}_{av_i} + |i - j| + \mathbf{D}_{v_j b}\}$$

$$= 2 \sum_{i=1}^{d} W(G_i) + 2 \sum_{1 \le i < j \le d} |V(G_i)||V(G_j)||i - j|$$

$$+ \sum_{i=1}^{d} (|V(G)| - |V(G_i)|) \sum_{a \in V(G_i)} \mathbf{D}_{av_i}$$

$$+ \sum_{j=1}^{d} (|V(G)| - |V(G_j)|) \sum_{b \in V(G_j)} \mathbf{D}_{bv_j}$$

$$= 2 \sum_{i=1}^{d} W(G_i) + 2 \sum_{1 \le i < j \le d} |V(G_i)||V(G_j)|(j - i)$$

$$+ 2 \sum_{i=1}^{d} (|V(G)| - |V(G_i)|) \sum_{a \in V(G_i)} \mathbf{D}_{av_i}.$$

Hence, by the definition of $P_v(G)$ we obtain that

$$W(G) = \sum_{i=1}^{d} W(G_i) + \sum_{1 \le i < j \le d} |V(G_i)||V(G_j)|(j-i)$$

$$+ \sum_{i=1}^{d} (|V(G)| - |V(G_i)|) P_{v_i}(G_i),$$

as claimed. □

The above theorem gives for $G_i = H$ and $v_i = v$ for all $i = 1, 2, \ldots, d$ the following result.

Corollary 3.2 *The Wiener index of the bridge graph $G = B(H, \ldots, H; v, \ldots, v)$ (d times) is given by*

$$W(G) = dW(H) + |V(H)|^2 \binom{d+1}{3} + (d-1)d|V(H)|P_v(H).$$

For large d one has asymptotically

$$W(G) \sim \frac{|V(H)|^2}{6} d^3.$$

3.2 Detour index of the bridge graph

Now we consider the detour index of the bridge graph.

Theorem 3.3 *The detour index of the bridge graph $G = B(G_1, \ldots, G_d; v_1, \ldots, v_d)$ is given by*

$$\omega(G) = \sum_{i=1}^{d} \omega(G_i) + \sum_{1 \le i < j \le d} |V(G_i)||V(G_j)| \left(j - i + c_{v_{i+1}} + \cdots + c_{v_{j-1}} \right)$$

$$+ \sum_{i=1}^{d} (|V(G)| - |V(G_i)|) Q_{v_i}(G_i),$$

where $c_{v_i} = c_{v_i}(G_i)$ is the length of the longest path in the graph G_i which starts and ends at v_i.

Proof Using the same arguments as in the proof of Theorem 3.1, we can write

$$2\omega(G) = 2\sum_{i=1}^{d} \omega(G_i) + 2 \sum_{1 \le i < j \le d} \sum_{a \in V(G_i)} \sum_{b \in V(G_j)} \Delta_{ab}.$$

Recalling Fact 2.1 about the decomposition of the longest path in a bridge graph, we can thus write

$$
\begin{aligned}
2\omega(G) = {} & 2\sum_{i=1}^{d}\omega(G_i) + 2\sum_{1\leq i<j\leq d}\sum_{a\in V(G_i)}\sum_{b\in V(G_j)}\Bigg(\boldsymbol{\Delta}_{av_i} + (j-i) \\
& + \sum_{s=i+1}^{j-1} c_{v_s} + \boldsymbol{\Delta}_{v_jb}\Bigg) \\
= {} & 2\sum_{i=1}^{d}\omega(G_i) + 2\sum_{1\leq i<j\leq d}|V(G_i)||V(G_j)|\left(j-i+c_{v_{i+1}}+\cdots+c_{v_{j-1}}\right) \\
& + 2\sum_{1\leq i<j\leq d}\sum_{a\in V(G_i)}\sum_{b\in V(G_j)}\left(\boldsymbol{\Delta}_{av_i}+\boldsymbol{\Delta}_{v_jb}\right) \\
= {} & 2\sum_{i=1}^{d}\omega(G_i) + 2\sum_{1\leq i<j\leq d}|V(G_i)||V(G_j)|\left(j-i+c_{v_{i+1}}+\cdots+c_{v_{j-1}}\right) \\
& + 2\sum_{i=1}^{d}(|V(G)| - |V(G_i)|)\sum_{a\in V(G_i)}\boldsymbol{\Delta}_{av_i}.
\end{aligned}
$$

Hence, by the definition of $Q_v(G)$ we obtain that

$$
\begin{aligned}
\omega(G) = {} & \sum_{i=1}^{d}\omega(G_i) + \sum_{1\leq i<j\leq d}|V(G_i)||V(G_j)|\left(j-i+c_{v_{i+1}}+\cdots+c_{v_{j-1}}\right) \\
& + \sum_{i=1}^{d}(|V(G)| - |V(G_i)|)Q_{v_i}(G_i),
\end{aligned}
$$

as claimed.   □

The above theorem gives for $G_i = H$ and $v_i = v$ for all $i = 1, 2, \ldots, d$ the following result.

Corollary 3.4 *The detour index of the bridge graph* $G = B(H, \ldots, H; v, \ldots, v)$ *(d times) is given by*

$$
\omega(G) = d\omega(H) + |V(H)|^2\binom{d+1}{3} + |V(H)|^2\binom{d}{3}c_v(H) + 2|V(H)|\binom{d}{2}Q_v(H),
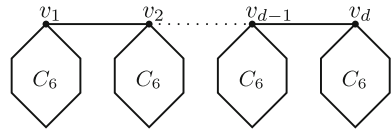$$

where $c_v(H)$ is the length of the longest path in the graph H which starts and ends at v. For large d one has asymptotically

$$
W(G) \sim \frac{|V(H)|^2(1+c_v(H))}{6}d^3.
$$

Fig. 3 The graph $B_{d;3}$

Fig. 4 The bridge graph of the hexagon graph C_6

3.3 Examples

In this section we consider some simple examples and determine their Wiener and detour indices.

Example 3.5 Let $B_{d;3} = G(P_3, \ldots, P_3; v, \ldots, v)$ (d times), where P_3 is the path with three vertices such that the middle vertex is v, see Fig. 3 (Polyethene when $d = 4$). Using

$$W(P_n) = \binom{n+1}{3},\qquad(3.1)$$

Corollary 3.2 yields for $B_{d;3}$ that

$$W(B_{d;3}) = 4d + 9\binom{d+1}{3} + 6(d-1)d = \frac{d(3d^2 + 12d - 7)}{2}.$$

Example 3.6 In a similar fashion one may define more general graphs $B_{d;n}$. Since one has $B_{2;n} = P_{2n}$ one may perform a simple consistency check. Using Corollary 3.2 as well as (3.1), one obtains after some calculations that $W(B_{2;n}) = \frac{n}{3}(4n^2 - 1)$ which coincides with $\binom{2n+1}{3} = W(P_{2n})$, as it should.

For the simple example $G = B_{d;3}$ Corollary 3.4 shows that $\omega(B_{d;3}) = W(B_{d;3})$. Let us, therefore, consider a more complex example for the detour index.

Example 3.7 Let us consider the bridge graph $G = (C_n, \ldots, C_n)$ of the cycle C_n with n vertices, see Fig. 4 for the case $n = 6$. We now want to determine the detour index $\omega(G)$. It is clear that $|V(C_n)| = n$ as well as $c_v(C_n) = n$, so it remains to determine $Q_v(C_n)$ and $\omega(C_n)$. Let us first turn to $Q_v(C_n)$ and consider the case where n is even. The vertex "opposite" to v contributes $\frac{n}{2}$ and the contributions from the vertices of the left and right side are equal. It follows that

$$\sum_{a \in V(C_n)} \Delta_{av} = \frac{n}{2} + 2\sum_{k=1}^{\frac{n}{2}-1}(n-k) = \frac{3n^2 - 4n}{4}.$$

In the case where n is odd one obtains by a similar consideration

$$\sum_{a \in V(C_n)} \Delta_{av} = \frac{3n^2 - 4n + 1}{4}.$$

To write this in a uniform fashion we introduce $\varepsilon_n := 0$ if n is even and $\varepsilon_n := 1$ if n is odd. Then we can write both cases simultaneously as

$$Q_v(C_n) = \sum_{a \in V(C_n)} \Delta_{av} = \frac{3n^2 - 4n + \varepsilon_n}{4}.$$

Due to the high symmetry of the graph C_n we can now easily determine $\omega(C_n)$. Since $\omega(C_n) = \frac{n}{2}Q_v(C_n)$ we obtain for the detour index of the cycle C_n that

$$\omega(C_n) = \frac{n(3n^2 - 4n + \varepsilon_n)}{8},$$

as was found originally by Lukovits [17] (see also [5]). Thus, collecting the above results we have found that the detour index of the bridge graph of d cycles C_n is given by

$$\omega(B(C_n, \ldots, C_n)) = \frac{n(3n^2 - 4n + \varepsilon_n)}{8}\left(d + 4\binom{d}{2}\right) + n^3\binom{d}{3} + n^2\binom{d+1}{3}.$$

4 Hyper-Wiener and hyper-detour indices of the bridge graph

In this section we give a formula for the hyper-Wiener and hyper-detour index of the bridge graph $B(G_1, G_2, \ldots, G_d)$ in terms of the graphs G_i.

4.1 Hyper-Wiener index of the bridge graph

At first we consider the hyper-Wiener index for the bridge graph.

Theorem 4.1 *The hyper-Wiener index of the bridge graph* $G = B(G_1, \ldots, G_d; v_1 \ldots, v_d)$ *is given by*

$$WW(G) = \sum_{i=1}^{d} WW(G_i) + \sum_{i=1}^{d}(|V(G)| - |V(G_i)|)\widetilde{P}_{v_i}(G_i)$$

$$+ \sum_{1 \leq i < j \leq d}\left(|V(G_i)||V(G_j)|\binom{j - i + 1}{2}\right.$$

$$+ 2(j - i)|V(G_j)|P_{v_i}(G_i) + P_{v_i}(G_i)P_{v_j}(G_j)\Big).$$

Proof Using similar arguments as in the proof of Theorem 3.1, we can write

$$2WW(G) = 2\sum_{i=1}^{d} WW(G_i) + 2 \sum_{1 \le i < j \le d} \sum_{a \in V(G_i)} \sum_{b \in V(G_j)} \binom{\mathbf{D}_{ab}+1}{2}.$$

Recalling Fact 2.1 about the decomposition of the shortest path in a bridge graph, we can thus write

$$2WW(G) = 2\sum_{i=1}^{d} WW(G_i) + 2 \sum_{1 \le i < j \le d} \sum_{a \in V(G_i)} \sum_{b \in V(G_j)} \binom{\mathbf{D}_{av_i}+(j-i)+\mathbf{D}_{v_jb}+1}{2}.$$

Using the fact that

$$\binom{\alpha + \beta + \gamma + 1}{2} = \binom{\alpha + 1}{2} + \binom{\beta + 1}{2} + \binom{\gamma + 1}{2} + \alpha\beta + \alpha\gamma + \beta\gamma, \quad (4.1)$$

we obtain that

$$2WW(G) = 2\sum_{i=1}^{d} WW(G_i) + 2\sum_{i=1}^{d}(|V(G)| - |V(G_i)|)\widetilde{P}_{v_i}(G_i)$$

$$+ 2 \sum_{1 \le i < j \le d} \left(|V(G_i)||V(G_j)| \binom{j-i+1}{2} \right.$$

$$+ 2(j-i)|V(G_j)|P_{v_i}(G_i) + P_{v_i}(G_i)P_{v_j}(G_j) \bigg),$$

as claimed.   □

The above theorem gives for $G_i = H$ and $v_i = v$ for all $i = 1, 2, \ldots, d$ the following result.

Corollary 4.2 *The hyper-Wiener index of the bridge graph* $G = B(H, \ldots, H; v, \ldots, v)$ *(d times) is given by*

$$WW(G) = dWW(H) + 2\binom{d}{2}|V(H)|\widetilde{P}_v(H)$$

$$+ \binom{d+2}{4}|V(H)|^2 + 2\binom{d+1}{3}|V(H)|P_v(H) + \binom{d}{2}P_v^2(H).$$

For large d one has asymptotically

$$WW(G) \sim \frac{|V(H)|^2}{24}d^4.$$

4.2 Hyper-detour index of the bridge graph

Now we consider the hyper-detour index of the bridge graph.

Theorem 4.3 *The hyper-detour index of the bridge graph* $G = B(G_1, \ldots, G_d; v_1, \ldots, v_d)$ *is given by*

$$
\omega\omega(G) = \sum_{i=1}^{d} \omega\omega(G_i) + \sum_{i=1}^{d} (|V(G)| - |V(G_i)|) \widetilde{Q}_{v_i}(G_i)
$$
$$
+ \sum_{1 \le i < j \le d} |V(G_i)||V(G_j)| \binom{j - i + \sum_{s=i+1}^{j-1} c_{v_s} + 1}{2}
$$
$$
+ \sum_{1 \le i < j \le d} \left[2 \left(j - i + \sum_{s=i+1}^{j-1} c_{v_s} \right) |V(G_j)| Q_{v_i}(G_i) + Q_{v_i}(G_i) Q_{v_j}(G_j) \right],
$$

where c_{v_i} is the length of the longest path in the graph G_i which starts and ends at v_i.

Proof Using similar arguments as in the proof of Theorem 3.1, we obtain that

$$
2\omega\omega(G) = 2 \sum_{i=1}^{d} \omega\omega(G_i)
$$
$$
+ 2 \sum_{1 \le i < j \le d} \sum_{a \in V(G_i)} \sum_{b \in V(G_j)} \binom{\mathbf{\Delta}_{ab} + 1}{2}.
$$

Recalling Fact 2.1 about the decomposition of the longest path in a bridge graph, we can thus write

$$
2\omega\omega(G) = 2 \sum_{i=1}^{d} \omega\omega(G_i)
$$
$$
+ 2 \sum_{1 \le i < j \le d} \sum_{a \in V(G_i)} \sum_{b \in V(G_j)}
$$
$$
\times \binom{\mathbf{\Delta}_{av_i} + (j - i) + \sum_{s=i+1}^{j-1} c_{v_s} + \mathbf{\Delta}_{v_j b} + 1}{2}.
$$

Using (4.1), we obtain that

$$2\omega\omega(G) = 2\sum_{i=1}^{d}\omega\omega(G_i) + 2\sum_{i=1}^{d}(|V(G)| - |V(G_i)|)\widetilde{Q}_{v_i}(G_i)$$

$$+ 2\sum_{1\leq i<j\leq d}|V(G_i)||V(G_j)|\binom{j-i+\sum_{s=i+1}^{j-1}c_{v_s}+1}{2}$$

$$+ 2\sum_{1\leq i<j\leq d}\left[2\left(j-i+\sum_{s=i+1}^{j-1}c_{v_s}\right)|V(G_j)|Q_{v_i}(G_i)+Q_{v_i}(G_i)Q_{v_j}(G_j)\right],$$

as claimed.   □

The above theorem gives for $G_i = H$ and $v_i = v$ for all $i = 1, 2, \ldots, d$ the following result.

Corollary 4.4 *The hyper-detour index of the bridge graph $G = B(H, \ldots, H; v, \ldots, v)$ (d times) is given by*

$$\omega\omega(G) = d\omega\omega(H) + 2|V(H)|\binom{d}{2}\widetilde{Q}_v(H)$$

$$+ |V(H)|^2 \sum_{1\leq i<j\leq d}\binom{j-i+1+(j-i-1)c_v(H)}{2}$$

$$+ Q_v(H)\sum_{1\leq i<j\leq d}[2\,(j-i+(j-i-1)c_v(H))\,|V(H)| + Q_v(H)],$$

where $c_v(H)$ is the length of the longest path in the graph H which starts and ends at v. For large d one has asymptotically

$$\omega\omega(G) \sim \frac{|V(H)|^2(1+c_v(H))^2}{24}d^4.$$

4.3 Examples

In this section we consider some simple examples.

Example 4.5 Using

$$WW(P_n) = \binom{n+2}{4}, \tag{4.2}$$

Corollary 4.2 yields for $B_{d;3}$ that

$$WW(B_{d;3}) = 5d + 12\binom{d}{2} + 9\binom{d+2}{4} + 12\binom{d+1}{3} + 4\binom{d}{2}$$
$$= \frac{d(3d^3 + 22d^2 + 61d - 46)}{8}.$$

Example 4.6 Recalling $B_{2;n} = P_{2n}$, one may perform a consistency check as in the case of the Wiener index (see Example 3.6). Inserting (4.2) into Corollary 4.2, one obtains after some calculations that $WW(B_{2;n}) = \frac{n}{6}(n+1)(4n^2 - 1)$ which coincides with $\binom{2n+2}{4} = WW(P_{2n})$, as it should.

Example 4.7 In the simple example $G = B_{d;3}$ as shown in Fig. 3 Corollary 4.4 yields that $\omega\omega(B_{d;3}) = WW(B_{d;3})$ (since $c_v(P_3) = 0$, $Q_v(P_3) = P_v(P_3)$ and $\widetilde{Q}_v(P_3) = \widetilde{P}_v(P_3)$).

5 Wiener and detour indices of the chain graph

In this section we give a formula for the Wiener and detour indices of the chain graph $C(G_1, G_2, \ldots, G_d)$ in terms of the graphs G_i. Before starting we introduce the following convenient notation

$$V_L(G_i) := V(G_1) \cup \cdots \cup V(G_{i-2}) \cup (V(G_{i-1}) \setminus \{v_i\}), \quad i = 2, \ldots, d$$

as well as

$$V_R(G_i) := (V(G_{i+1}) \setminus \{w_i\}) \cup V(G_{i+2}) \cup \cdots \cup V(G_d), \quad i = 1, \ldots d - 1.$$

Clearly, $V_L(G_i)$ denotes the set of vertices to the left of G_i and $V_R(G_i)$ denotes the set of vertices to the right of G_i.

5.1 Wiener index of the chain graph

At first we consider the Wiener index for the chain graph.

Theorem 5.1 *The Wiener index of the chain graph $G = C(G_1, \ldots, G_d; v_1, w_1, \ldots, v_d, w_d)$ is given by*

$$W(G) = \sum_{i=1}^{d} W(G_i) + \left(R_P(G_1) + \sum_{s=2}^{d-1} \{L_P(G_s) + R_P(G_s)\} + L_P(G_d) \right)$$

$$+ \frac{1}{2} \sum_{i=2}^{d} \left(\sum_{j=2}^{i} (|V(G_i)| - 1)(|V(G_{j-1})| - 1)\mathbf{D}_{v_j v_i} \right.$$

$$\left. + \sum_{j=i}^{d} (|V(G_{i-1})| - 1)(|V(G_j)| - 1)\mathbf{D}_{v_i v_j} \right).$$

Proof Similar arguments as in the proof of Theorem 3.1 yield that

$$2W(G) = 2\sum_{i=1}^{d} W(G_i) + \sum_{i=2}^{d} \sum_{j=1}^{i-1} \sum_{a \in V_L(G_i)} \sum_{b \in V_R(G_j)} \mathbf{D}_{ab}$$

$$+ \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \sum_{a \in V_R(G_i)} \sum_{b \in V_L(G_j)} \mathbf{D}_{ab}.$$

Recalling Fact 2.2 about the decomposition of the shortest path in a chain graph, we can thus write

$$2W(G) = 2\sum_{i=1}^{d} W(G_i) + \sum_{i=2}^{d} \sum_{j=1}^{i-1} \sum_{a \in V_L(G_i)} \sum_{b \in V_R(G_j)} \left(\mathbf{D}_{av_i} + \sum_{s=j+1}^{i-1} \mathbf{D}_{v_s w_s} + \mathbf{D}_{bw_j} \right)$$

$$+ \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \sum_{a \in V_R(G_i)} \sum_{b \in V_L(G_j)} \left(\mathbf{D}_{aw_i} + \sum_{s=i+1}^{j-1} \mathbf{D}_{v_s w_s} + \mathbf{D}_{v_j b} \right).$$

Let us consider first

$$T_1 := \sum_{i=2}^{d} \sum_{j=1}^{i-1} \sum_{a \in V_L(G_i)} \sum_{b \in V_R(G_j)} (\mathbf{D}_{av_i} + \mathbf{D}_{bw_j})$$

$$+ \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \sum_{a \in V_R(G_i)} \sum_{b \in V_L(G_j)} (\mathbf{D}_{aw_i} + \mathbf{D}_{v_j b}).$$

This is equal to

$$\sum_{i=1}^{d} \sum_{j=1}^{i-1} (|V(G_j)| - 1)P_{v_i}(G_i) + \sum_{i=1}^{d} \sum_{j=1}^{i-1} (|V(G_i) - 1)|P_{w_j}(G_j)$$

$$+ \sum_{i=1}^{d} \sum_{j=i+1}^{d} (|V(G_j)| - 1)P_{w_i}(G_i) + \sum_{i=1}^{d} \sum_{j=i+1}^{d} (|V(G_i)| - 1)P_{v_j}(G_j).$$

Considering the first and fourth sum yields

$$\sum_{i=1}^{d}\sum_{j=1}^{i-1}(|V(G_j)|-1)P_{v_i}(G_i)+\sum_{j=1}^{d}\sum_{i=j+1}^{d}(|V(G_j)|-1)P_{v_i}(G_i)$$

$$=2\sum_{i=2}^{d}P_{v_i}(G_i)\sum_{j=1}^{i-1}(|V(G_j)|-1).$$

Considering the second and third sum yields in a similar fashion

$$\sum_{i=1}^{d}\sum_{j=1}^{i-1}(|V(G_i)|-1)P_{w_j}(G_j)+\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}(|V(G_j)|-1)P_{w_i}(G_i)$$

$$=2\sum_{i=1}^{d-1}P_{v_{i+1}}(G_i)\sum_{j=i+1}^{d}(|V(G_j)|-1).$$

Thus, in total we obtain

$$T_1=2\left(\sum_{i=2}^{d}P_{v_i}(G_i)\sum_{j=1}^{i-1}(|V(G_j)|-1)+\sum_{i=1}^{d-1}P_{v_{i+1}}(G_i)\sum_{j=i+1}^{d}(|V(G_j)|-1)\right).$$

Recalling the definition of $L_P(G_i)$ and $R_P(G_i)$, this can be written as

$$T_1=2\left(\sum_{i=2}^{d}L_P(G_i)+\sum_{i=1}^{d-1}R_P(G_i)\right)$$

$$=2\left(R_P(G_1)+\sum_{s=2}^{d-1}\{L_P(G_s)+R_P(G_s)\}+L_P(G_d)\right).$$

Now, let us consider the remaining sums, i.e.,

$$T_2:=\sum_{i=2}^{d}\sum_{j=1}^{i-1}\sum_{a\in V_L(G_i)}\sum_{b\in V_R(G_j)}\sum_{s=j+1}^{i-1}\mathbf{D}_{v_s w_s}$$

$$+\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}\sum_{a\in V_R(G_i)}\sum_{b\in V_L(G_j)}\sum_{s=i+1}^{j-1}\mathbf{D}_{v_s w_s}$$

$$=\sum_{i=2}^{d}\sum_{j=2}^{i}(|V(G_i)|-1)(|V(G_{j-1})|-1)\mathbf{D}_{v_j v_i}$$

$$+\sum_{i=2}^{d}\sum_{j=i}^{d}(|V(G_{i-1})|-1)(|V(G_j)|-1)\mathbf{D}_{v_i v_j},$$

where we have used in the last line that $\sum_{s=j+1}^{i-1} \mathbf{D}_{v_s w_s} = \mathbf{D}_{v_{j+1} w_{i-1}}$ and $w_{i-1} = v_i$.

It remains to collect the above results. From the definition of T_1 and T_2 it follows that

$$
\begin{aligned}
W(G) &= \sum_{i=1}^{d} W(G_i) + \frac{1}{2} T_1 + \frac{1}{2} T_2 \\
&= \sum_{i=1}^{d} W(G_i) + \left(R_P(G_1) + \sum_{s=2}^{d-1} \{ L_P(G_s) + R_P(G_s) \} + L_P(G_d) \right) \\
&\quad + \frac{1}{2} \sum_{i=2}^{d} \left(\sum_{j=2}^{i} (|V(G_i)| - 1)(|V(G_{j-1})| - 1) \mathbf{D}_{v_j v_i} \right. \\
&\quad \left. + \sum_{j=i}^{d} (|V(G_{i-1})| - 1)(|V(G_j)| - 1) \mathbf{D}_{v_i v_j} \right),
\end{aligned}
$$

as claimed. $\square$

As a corollary to the above theorem we have the following result.

Corollary 5.2 *The Wiener index of the chain graph $G = C(H, \ldots, H; v, w, \ldots, v, w)$ (d times) is given by*

$$
W(G) = d W(H) + (|V(H)| - 1)(P_v(H) + P_w(H)) \binom{d}{2}
$$

$$
+ (|V(H)| - 1)^2 \mathbf{D}_{vw} \binom{d}{3}.
$$

For large d one has asymptotically

$$
W(G) \sim \frac{(|V(H)| - 1)^2 \mathbf{D}_{vw}}{6} d^3.
$$

Proof From Theorem 5.1 we directly obtain

$$
\begin{aligned}
W(G) &= d W(H) + \left\{ \sum_{i=2}^{d} L_P(H) + \sum_{i=1}^{d-1} R_P(H) \right\} \\
&\quad + \frac{(V(H) - 1)^2}{2} \sum_{i=2}^{d} \left(\sum_{j=2}^{i} \mathbf{D}_{v_j v_i} + \sum_{j=i}^{d} \mathbf{D}_{v_i v_j} \right),
\end{aligned}
$$

where we have used in the last line that

$$
\mathbf{D}_{v_j v_i} + \mathbf{D}_{v_i v_j} = \sum_{s=j}^{i-1} \mathbf{D}_{v_s v_{s+1}} = \mathbf{D}_{vw}(i - j).
$$

Recalling (2.3), we find

$$R_P(H) = P_w(H)(|V(H)| - 1)(d - i), \quad L_P(H) = P_v(H)(|V(H)| - 1)(i - 1),$$

showing that the term in the above bracket equals

$$\sum_{i=2}^{d} L_P(H) + \sum_{i=1}^{d-1} R_P(H) = (|V(H)| - 1)$$

$$\times \left(P_v(H) \sum_{i=2}^{d} (i - 1) + P_w(H) \sum_{i=1}^{d-1} (d - i) \right)$$

$$= (|V(H)| - 1)(P_v(H) + P_w(H)) \binom{d}{2}.$$

It follows that

$$W(G) = dW(H) + (|V(H)| - 1)(P_v(H) + P_w(H)) \binom{d}{2} + (|V(H)| - 1)^2 \mathbf{D}_{vw} \binom{d}{3},$$

as requested.   □

5.2 Detour index of the chain graph

The case of detour index of the chain graph is very similar to the case of Wiener index of the chain graph; the reason is that the expressions for the shortest and longest path are very similar (see Fact 2.2), in contrast to the case of the bridge graph (see Fact 2.1). A minor modification of the proof of Theorem 5.1 (replacing everywhere D_{av} by $\mathbf{\Delta}_{av}$) leads to the following result.

Theorem 5.3 *The detour index of the chain graph* $G = C(G_1, \ldots, G_d; v_1, w_1, \ldots, v_d, w_d)$ *is given by*

$$\omega(G) = \sum_{i=1}^{d} \omega(G_i) + \left(R_Q(G_1) + \sum_{i=2}^{d-1} \{ L_Q(G_i) + R_Q(G_i) \} + L_Q(G_d) \right)$$

$$+ \frac{1}{2} \sum_{i=2}^{d} \left(\sum_{j=2}^{i} (|V(G_i)| - 1)(|V(G_{j-1})| - 1) \mathbf{\Delta}_{v_j v_i} \right.$$

$$\left. + \sum_{j=i}^{d} (|V(G_{i-1})| - 1)(|V(G_j)| - 1) \mathbf{\Delta}_{v_i v_j} \right).$$

In the same fashion as for the Wiener index we obtain as a corollary to the above theorem the following result.

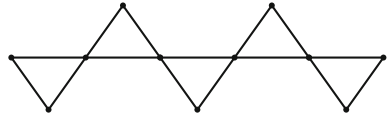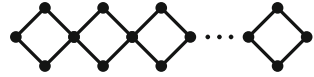**Fig. 5** The spiro-chain of C_3
(with $d = 5$)

Fig. 6 The spiro-chain of
$C_4(1, 3)$

Corollary 5.4 *The detour index of the chain graph $G = C(H, \ldots, H; v, w, \ldots, v, w)$*
(d times) is given by

$$\omega(G) = d\omega(H) + (|V(H)|-1)(Q_v(H) + Q_w(H))\binom{d}{2} + (|V(H)| - 1)^2 \mathbf{\Delta}_{vw}\binom{d}{3}.$$

For large d one has asymptotically

$$\omega(G) \sim \frac{(|V(H)| - 1)^2 \mathbf{\Delta}_{vw}}{6} d^3.$$

5.3 Examples

In this section we consider three particular examples of *spiro-graphs* which are also
examples for chain graphs. In particular, we choose examples where $G_i = H$ for all i.
The notation follows closely [5]; we denote the spiro-chain containing the graph H d
times with $\mathcal{S}_d(H)$. Note that the Wiener indices of the examples considered here can
be compared with the results given in [5] (which are based on MAPLE).

Example 5.5 The spiro-chain of the cycle C_3 is given in Fig. 5. Since $W(C_3) =$
3, $D_{vw} = 1$, $P_v(C_3) = P_w(C_3) = 2$, application of Corollary 5.2 yields for the
Wiener index that

$$W(\mathcal{S}_d(C_3)) = 3d + 2 \cdot 4 \cdot \frac{d(d - 1)}{2} + 4 \cdot 1 \cdot \frac{d(d - 1)(d - 2)}{2 \cdot 3} = \frac{d}{3}(2d^2 + 6d + 1),$$
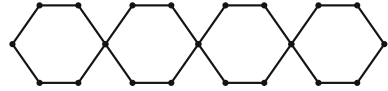
as given in [5]. Similarly, using $\omega(C_3) = 6$, $\mathbf{\Delta}_{vw} = 2$, $Q_v(C_3) = Q_w(C_3) = 4$,
application of Corollary 5.4 yields for the detour index

$$\omega(\mathcal{S}_d(C_3)) = 6d + 2 \cdot 8 \cdot \frac{d(d-1)}{2} + 4 \cdot 2 \cdot \frac{d(d - 1)(d - 2)}{2 \cdot 3} = \frac{d}{3}(4d^2 - 4d + 18).$$

Example 5.6 The spiro-chain of the square $C_4(1, 3)$ is given in Fig. 6. Since
$W(C_4(1, 3)) = 8$, $P_v(C_4(1, 3)) = P_w(C_4(1, 3)) = 4$, $\mathbf{D}_{vw} = 2$, application of Cor-
ollary 5.2 yields for the Wiener index that

$$W(\mathcal{S}_d(C_4(1, 3))) = 8d + 12d(d - 1) + 3d(d - 1)(d - 2) = d(3d^2 + 3d + 2),$$

as given in [5]. Similarly, using $\omega(C_4(1, 3)) = 16, \Delta_{vw} = 2, Q_v(C_4(1, 3)) = Q_w(C_4(1, 3)) = 8$, application of Corollary 5.4 yields for the detour index

$$\omega(\mathcal{S}_d(C_4(1, 3))) = 16d + 24d(d - 1) + 3d(d - 1)(d - 2) = d(3d^2 + 15d - 2).$$

Example 5.7 The spiro-chain of the square $C_6(1, 4)$ is given in Fig. 7. Since $W(C_6(1, 4)) = 27, P_v(C_6(1, 4)) = P_w(C_6(1, 4)) = 9, \mathbf{D}_{vw} = 3$, application of Corollary 5.2 yields for the Wiener index that

$$W(\mathcal{S}_d(C_6(1, 4))) = \frac{d}{2}(25d^2 + 15d + 14),$$

as given in [5]. Similarly, using $\omega(C_6(1, 4)) = 63, \Delta_{vw} = 3, Q_v(C_6(1, 4)) = Q_w(C_6(1, 4)) = 21$, application of Corollary 5.4 yields for the detour index

$$\omega(\mathcal{S}_d(C_6(1, 4))) = \frac{d}{2}(25d^2 + 135d - 34).$$

In the next examples we show how to generalize the last example to the spiro-chain where the vertices v and w are not "opposite". In contrast to the above examples these examples will not be pursued further in later sections, although it would be straightforward to do so.

Example 5.8 In the last example we have treated the spiro-chain of the square $C_6(1, 4)$. It is clear that given a fixed numbering of the underlying graph C_6 the vertices v and w can be arbitrary (but different) numbers between 1 and 6. Choosing the numbering such that vertex v has number 1, the number i of the vertex w has to be in $\{2, 3, 4, 5, 6\}$. However, due to the symmetry $k \leftrightarrow 6 - k + 2$ one can restrict i to $\{2, 3, 4\}$. Thus, denoting the graph where the vertices v and w have numbers k and l by $C_6(k, l)$, we can restrict to the cases $C_6(1, i)$ with $i \in \{2, 3, 4\}$. To calculate the Wiener index we note that the only difference to above is that now $\mathbf{D}_{vw} = i - 1$ (and also $W(C_6(1, i)) = 27, P_v(C_6(1, i)) = P_w(C_6(1, i)) = 9$). Thus, application of Corollary 5.2 yields for the Wiener index that

$$W(\mathcal{S}_d(C_6(1, i))) = \frac{d}{6}\left\{25(i - 1)d^2 + (270 - 75(i - 1))d + 50i - 158\right\}.$$

Choosing $i = 4$ yields the formula given above, whereas $W(\mathcal{S}_d(C_6(1, 2))) = \frac{d}{6}(25d^2 + 195d - 58)$ as well as $W(\mathcal{S}_d(C_6(1, 2))) = \frac{d}{3}(25d^2 + 60d - 4)$, as given in [5]. Similarly, using $\Delta_{vw} = 7 - i$ (and also $\omega(C_6(1, i)) = 63, Q_v(C_6(1, i)) = Q_w(C_6(1, i)) = 21$), application of Corollary 5.4 yields for the detour index

$$\omega(\mathcal{S}_d(C_6(1,i))) = \frac{d}{6}\left\{25(7-i)d^2 + (630 - 75(7-i))d + 98 - 50i\right\}.$$

Example 5.9 In a similar fashion one can consider the case of C_N with arbitrary N. We only consider the case where N is even, i.e., $N = 2k$. Then we can consider the family $C_{2k}(1,i)$ with $i \in \{2,3,\ldots,k+1\}$. Using $\mathbf{D}_{vw} = i-1$ (and also $W(C_{2k}(1,i)) = k^3$, $P_v(C_{2k}(1,i)) = P_w(C_{2k}(1,i)) = k^2$) application of Corollary 5.2 yields for the Wiener index that

$$W(\mathcal{S}_d(C_{2k}(1,i))) = \frac{d}{6}\left\{ (2k-1)^2(i-1)d^2 + (2k-1)\left[6k^2 - 3(2k-1)(i-1)\right]d \right.$$
$$\left. +2(2k-1)^2(i-1) - 6k^2(k-1)\right\}.$$

6 Hyper-Wiener and hyper-detour indices of the chain graph

In this section we give a formula for the hyper-Wiener and hyper-detour indices of the chain graph $C(G_1, G_2, \ldots, G_d)$ in terms of the graphs G_i.

6.1 Hyper-Wiener index of the chain graph

At first we consider the hyper-Wiener index of the chain graph.

Theorem 6.1 *The hyper-Wiener index of the chain graph* $G = C(G_1, \ldots, G_d; v_1, w_1, \ldots, v_d, w_d)$ *is given by*

$$WW(G) = \sum_{i=1}^{d} WW(G_i) + \left(\widetilde{R}_P(G_1) + \sum_{s=2}^{d-1}\{\widetilde{L}_P(G_s) + \widetilde{R}_P(G_s)\} + \widetilde{L}_P(G_d)\right)$$
$$+ \sum_{1\leq j<i\leq d} (|V(G_i)|-1)(|V(G_j)|-1)\binom{\mathbf{D}_{v_{j+1}v_i}+1}{2}$$
$$+ \sum_{1\leq j<i\leq d} (|V(G_i)|-1)P_{w_j}(G_j)\mathbf{D}_{v_{j+1}v_i}$$
$$+ \sum_{1\leq j<i\leq d} (|V(G_j)|-1)P_{v_i}(G_i)\mathbf{D}_{v_{j+1}v_i}$$
$$+ \sum_{1\leq j<i\leq d} P_{v_i}(G_i)P_{w_j}(G_j).$$

Proof Using similar arguments as in the proof of Theorem 5.1, we obtain that

$$WW(G) = \sum_{i=1}^{d} WW(G_i) + \sum_{1\leq j<i\leq d}\ \sum_{a\in V_L(G_i)}\ \sum_{b\in V_R(G_j)} \binom{\mathbf{D}_{ab}+1}{2}.$$

Recalling Fact 2.2 about the decomposition of the shortest path in a chain graph, we can thus write $\mathbf{D}_{ab} = \mathbf{D}_{av_i} + \sum_{s=j+1}^{i-1} \mathbf{D}_{v_s w_s} + \mathbf{D}_{bw_j}$. Using the fact that $\sum_{s=j+1}^{i-1} \mathbf{D}_{v_s w_s} = \mathbf{D}_{v_{j+1}w_{i-1}}$ and $w_{i-1} = v_i$, we get that

$$
WW(G) = \sum_{i=1}^{d} WW(G_i)
$$
$$
+ \sum_{1 \le j < i \le d} \sum_{a \in V_L(G_i)} \sum_{b \in V_R(G_j)} \binom{\mathbf{D}_{av_i} + \mathbf{D}_{v_{j+1}v_i} + \mathbf{D}_{bw_j} + 1}{2}.
$$

Therefore, by (4.1) we have that

$$
WW(G) = \sum_{i=1}^{d} WW(G_i) + \sum_{i=2}^{d} \widetilde{P}_{v_i}(G_i) \sum_{j=1}^{i-1} (|V(G_j)| - 1)
$$
$$
+ \sum_{j=1}^{d-1} \widetilde{P}_{w_j}(G_j) \sum_{i=j+1}^{d} (|V(G_i)| - 1)
$$
$$
+ \sum_{1 \le j < i \le d} (|V(G_i)| - 1)(|V(G_j)| - 1) \binom{\mathbf{D}_{v_{j+1}v_i} + 1}{2}
$$
$$
+ \sum_{1 \le j < i \le d} (|V(G_i)| - 1) P_{w_j}(G_j) \mathbf{D}_{v_{j+1}v_i}
$$
$$
+ \sum_{1 \le j < i \le d} (|V(G_j)| - 1) P_{v_i}(G_i) \mathbf{D}_{v_{j+1}v_i}
$$
$$
+ \sum_{1 \le j < i \le d} P_{v_i}(G_i) P_{w_j}(G_j).
$$

Recalling the definition of $\widetilde{L}_P(G_i)$ and $\widetilde{R}_P(G_i)$, this is equivalent to our claim.   □

Using Theorem 6.1, a similar proof to Corollary 5.4 yields that

Corollary 6.2 *The hyper-Wiener index of the chain graph* $G = C(H, \ldots, H; v, w, \ldots, v, w)$ *(d times) is given by*

$$
WW(G) = dWW(H) + (|V(H)| - 1)(\widetilde{P}_v(H) + \widetilde{P}_w(H)) \binom{d}{2}
$$
$$
+ (|V(H)| - 1)^2 \binom{d}{3} \left(\frac{\mathbf{D}_{vw}(d-1) + 2}{4} \right) \mathbf{D}_{vw}
$$
$$
+ (|(V(H)| - 1)(P_v(H) + P_w(H)) \mathbf{D}_{vw} \binom{d}{3} + P_v(H) P_w(H) \binom{d}{2}.
$$

For large d one has asymptotically

$$WW(G) \sim \frac{(|V(H)| - 1)^2 \mathbf{D}_{vw}^2}{24} d^4.$$

6.2 Hyper-detour index of the chain graph

The case of hyper-detour index of the chain graph is very similar to the case of the hyper-Wiener index of the chain graph. A minor modification of the proof of Theorem 6.1 yields the following result.

Theorem 6.3 *The hyper-detour index of the chain graph* $G = C(G_1, \ldots, G_d;$ $v_1, w_1, \ldots, v_d, w_d)$ *is given by*

$$\omega\omega(G) = \sum_{i=1}^{d} \omega\omega(G_i) + \left(\tilde{R}_Q(G_1) + \sum_{s=2}^{d-1} \{\tilde{L}_Q(G_s) + \tilde{R}_P(G_s)\} + \tilde{L}_Q(G_d) \right)$$

$$+ \sum_{1 \le j < i \le d} (|V(G_i)| - 1)(|V(G_j)| - 1)\binom{\mathbf{\Delta}_{v_{j+1}v_i} + 1}{2}$$

$$+ \sum_{1 \le j < i \le d} (|V(G_i)| - 1)Q_{w_j}(G_j)\mathbf{\Delta}_{v_{j+1}v_i}$$

$$+ \sum_{1 \le j < i \le d} (|V(G_j)| - 1)Q_{v_i}(G_i)\mathbf{\Delta}_{v_{j+1}v_i}$$

$$+ \sum_{1 \le j < i \le d} Q_{v_i}(G_i)Q_{w_j}(G_j).$$

Using Theorem 6.3, a similar proof to Corollary 5.4 yields that

Corollary 6.4 *The hyper-detour index of the chain graph* $G = C(H, \ldots, H;$ $v, w, \ldots, v, w)$ *(d times) is given by*

$$\omega\omega(G) = d\omega\omega(H) + (|V(H)| - 1)(\tilde{Q}_v(H) + \tilde{Q}_w(H))\binom{d}{2}$$

$$+ (|V(H)| - 1)^2 \binom{d}{3}\left(\frac{\mathbf{\Delta}_{vw}(d-1) + 2}{4}\right)\mathbf{\Delta}_{vw}$$

$$+ (|V(H)| - 1)(Q_v(H) + Q_w(H))\mathbf{\Delta}_{vw}\binom{d}{3} + Q_v(H)Q_w(H)\binom{d}{2}.$$

For large d one has asymptotically

$$\omega\omega(G) \sim \frac{(|V(H)| - 1)^2 \mathbf{\Delta}_{vw}^2}{24} d^4.$$

6.3 Examples

In this section we continue the study of the spiro-graphs introduced in Sect. 5.3 and compute their hyper-Wiener and hyper-detour indices. Note that the hyper-Wiener indices of the examples considered here can be compared with the results given in [5] (which are based on MAPLE).

Example 6.5 The spiro-chain of the cycle C_3 is given in Fig. 5. Since $WW(C_3) = 3$, $D_{vw} = 1$, $\widetilde{P}_v(C_3) = \widetilde{P}_w(C_3) = 2$, $P_v(C_3) = P_w(C_3) = 2$, application of Corollary 6.2 yields for the hyper-Wiener index that

$$WW(\mathcal{S}_d(C_3)) = \frac{d^2}{6}(d^2 + 6d + 11),$$

as given in [5]. Similarly, using $\omega\omega(C_3) = 9$, $\Delta_{vw} = 2$, $\widetilde{Q}_v(C_3) = \widetilde{Q}_w(C_3) = 6$, $Q_v(C_3) = Q_w(C_3) = 4$, application of Corollary 6.4 yields for the hyper-detour index

$$\omega\omega(\mathcal{S}_d(C_3)) = \frac{d}{3}(2d^3 + 10d^2 + 27d - 12).$$

Example 6.6 The spiro-chain of the square $C_4(1, 3)$ is given in Fig. 6. Since $WW(C_4(1, 3)) = 10$, $\widetilde{P}_v(C_4(1, 3)) = \widetilde{P}_w(C_4(1, 3)) = 5$, $P_v(C_4(1, 3)) = P_w(C_4(1, 3)) = 4$, $D_{vw} = 2$, application of Corollary 6.2 yields for the hyper-Wiener index that

$$WW(\mathcal{S}_d(C_4(1, 3))) = \frac{d}{2}(3d^3 + 7d^2 + 4d + 6),$$

as given in [5]. Similarly, using $\omega\omega(C_4(1, 3)) = 30$, $\Delta_{vw} = 2$, $\widetilde{Q}_v(C_4(1, 3)) = \widetilde{Q}_w(C_4(1, 3)) = 15$, $Q_v(C_4(1, 3)) = Q_w(C_4(1, 3)) = 8$, application of Corollary 6.4 yields for the hyper-detour index

$$\omega\omega(\mathcal{S}_d(C_4(1, 3))) = \frac{d}{2}(3d^3 + 23d^2 + 64d - 30).$$

Example 6.7 The spiro-chain of $C_6(1, 4)$ is given in Fig. 7. Since $WW(C_6(1, 4)) = 42$, $\widetilde{P}_v(C_6(1, 4)) = \widetilde{P}_w(C_6(1, 4)) = 14$, $P_v(C_6(1, 4)) = P_w(C_6(1, 4)) = 9$, $D_{vw} = 3$, application of Corollary 6.2 yields for the hyper-Wiener index that

$$WW(\mathcal{S}_d(C_6(1, 4))) = \frac{d}{8}(75d^3 + 110d^2 + 29d + 122),$$

as given in [5]. Similarly, using $\omega\omega(C_6(1, 4)) = 1278$, $\Delta_{vw} = 3$, $\widetilde{Q}_v(C_6(1, 4)) = \widetilde{Q}_w(C_6(1, 4)) = 426$, $Q_v(C_6(1, 4)) = Q_w(C_6(1, 4)) = 21$, application of Corollary 6.4 yields for the hyper-detour index

$$\omega\omega(\mathcal{S}_d(C_6(1, 4))) = \frac{d}{8}(75d^3 + 590d^2 + 16509d - 6950).$$

Let us collect the results about the examples in the following table.

Index	$\mathcal{S}_d(C_3)$	$\mathcal{S}_d(C_4(1,3))$	$\mathcal{S}_d(C_6(1,4))$
Wiener	$\frac{d}{3}(2d^2 + 6d + 1)$	$d(3d^2 + 3d + 2)$	$\frac{d}{2}(25d^2 + 15d + 14)$
Detour	$\frac{d}{3}(4d^2 - 4d + 18)$	$d(3d^2 + 15d - 2)$	$\frac{d}{2}(25d^2 + 135d - 34)$
Hyper-Wiener	$\frac{d^2}{6}(d^2 + 6d + 11)$	$\frac{d}{2}(3d^3 + 7d^2 + 4d + 6)$	$\frac{d}{8}(75d^3 + 110d^2 + 29d + 122)$
Hyper-detour	$\frac{d}{3}(2d^3 + 10d^2 + 27d - 12)$	$\frac{d}{2}(3d^3 + 23d^2 + 64d - 30)$	$\frac{d}{8}(75d^3 + 590d^2 + 16509d - 6950)$

7 Conclusion

In this paper we have derived the Wiener, hyper-Wiener, detour and hyper-detour indices of bridge and chain graphs. The important special case where the bridge and chain graphs are built from several copies of the same graph (and connected in the same fashion, i.e., via the same vertices) was singled out and the asymptotics of the above indices for large graphs were given in this case. Furthermore, some explicit examples were treated in detail, in particular some spiro-graphs as examples of chain graphs built from copies of the same graph H connected via the same vertices. However, as the general formulas show it is quite cumbersome to determine the above indices for arbitrary chain graphs and this is true already for the case where the graph H is the same, but the connecting vertices are different ones. Clearly, the indices considered in the present paper are among the best known ones but there do exist many other different topological indices which have been considered in the (mathematical or chemical) literature. The consideration of some of the other indices for bridge and chain graphs is left to future investigations. As mentioned in the introduction, the Wiener index was determined in [2] for a class of graphs containing in particular the bridge and chain graphs. This suggests a common generalization of the results derived in [2] and the present article, namely to determine the hyper-Wiener, detour and hyper-detour indices for the class of graphs considered in [2].

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